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The propagator of the Calogero–Moser system in an external quadratic potential

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Abstract

We obtain the Hamilton operator of the Calogero–Moser quantum system in an external quadratic potential by conjugating the radial part for the action of $SO(n)$ by conjugacy of the Hamilton operator of the quantum harmonic oscillator on the Euclidean vector space of real symmetric matrices. Then, with Mehler's formula, we derive the propagator of the problem. We also investigate some schemes to change the interaction constant. For two-particle systems, we obtain explicit formulae, whereas for many-particle systems, we reduce the computation of the propagator to finding a definite integral. We give also the short time approximation, the energy levels and the trace of the propagation operator. © 1998 Published by Elsevier Science B.V.

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1. Introduction

Few quantum mechanics problems are exactly solvable. Except some one-body problems, the only known solvable problems are those the classical-mechanics counterparts of which are completely integrable: Toda lattices and Calogero–Moser systems [2] or similar ones, such as Sutherland's [19,20].

The general idea of this paper comes from a description of the rational Calogero model with an external quadratic potential as a projection of a harmonic motion in a matrix space. This idea has been used first for the explicit integration of the equations of motion in the

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classical case in papers by Olshaneski and Perelomov [13,14]. Kazdhan et al. [10] proved that this projection method coincides with the Marsden–Weinstein symplectic reduction scheme (see also [17] and Françoise [4], who has solved a conjecture made by Gallavoti and Marchioro [6]).

This idea is also very useful in the quantum case and sometimes gives the integral representations for wave functions and propagators (Green functions). This is emphasized in the review paper of Olshaneski and Perelomov [16], where an analogue of Theorem 4 of this paper is given for more general cases.

Olshaneski and Perelomov [15], by generalizing some results of Berezin [1], have noticed that the radial part of the Laplace–Beltrami operator of symmetric spaces is conjugated to the quantum Hamilton operator of the Sutherland system, which describes the motion of identical particles on a circle or a branch of a hyperbola with a pairwise interaction potential inversely proportional to the square of their mutual distance. They also observed that the problem on the line (Calogero's problem) was obtained as a limit when the curvature of the symmetric space goes to zero.

In this paper, we consider the problem on the line, starting not from a symmetric space, but from the space of real symmetric matrices, which can be understood as the limit of the symmetric space $U(n)/SO(n)$ when its curvature goes to zero. Thus, taking the radial part of the Laplace–Beltrami operator for the action of $SO(n)$, we directly obtain the problem on the line. This scheme allows us to add an external quadratic potential before reduction, that we find again in the reduced one.

The problem thus obtained is slightly different from the one Calogero has studied [2], where the quadratic potential is pairwise rather than external.

In Section 2, we fix the notations in use along this paper. In Section 3, we give the radial reduction leading to Calogero's problem. Section 4 is devoted to the calculation of the propagator giving the time evolution in terms of the initial wave function, by use of methods analogous to the one given by Debiard and Gaveau [3]. In Section 5, we give the short time approximation of the propagator by the stationary phase method. This approximation is not exact, but becomes so when the interaction potential vanishes, which allows to compute the Fourier transform of the delta function of adjoint orbits in $u(n)$. Section 6 shows how other constants in the $1/r^2$ pairwise potential may be obtained and the corresponding propagators are computed therein. The obtention of systems by reduction allows us to determine the eigenvalues of the Hamilton operator, and then to compute the trace of the propagation operator, which is done in Section 7. The results we obtain deal with the propagation in a Weyl chamber. In Section 8, we derive the physical propagation according to the statistics the particles obey to.

2. Notations

The space of $n \times n$ real symmetric matrices will be denoted by V, and will be provided with the scalar product: $\langle X, Y\rangle = tr(XY)$. The group $SO(n)$ operates on V by conjugacy: $(g, X) \mapsto gXg^{-1}$ ($g \in SO(n)$, $X \in V$) and preserves the scalar product on V.

The submanifold

$$\Lambda = \left\{ \begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & \ddots & \\ & & & h_n \end{pmatrix}, h_1 < h_2 < \cdots < h_n \right\}$$

is transversal to the action of $SO(n)$ and orthogonal to the orbits since the tangent space to the orbit through $H \in \Lambda$ is $\{[Z, H] = ZH - HZ | Z$ skewsymmetric$\}$, the tangent space to Λ is the space of diagonal matrices H' and

$$\langle H', [Z, H]\rangle = tr(H'ZH - H'HZ) = 0 \quad \text{because } HH' = H'H.$$

We denote by

$$M = \left\{ \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}, \text{with an even number of } -1 \right\}$$

the isotropy subgroup in $SO(n)$ of matrices in Λ.

We provide $SO(n)$ with the bi-invariant metric induced by the invariant scalar product on the Lie-algebra $SO(n) : (X, Y) \mapsto -tr(XY)$.

We will denote by $\mathrm{d}g$ and $\mathrm{d}\bar{g}$ the corresponding invariant measures on $SO(n)$ and $SO(n)/M$, respectively.

We will also use the density function δ, which is the ratio between the Riemannian measure induced by V on an orbit and the measure $\mathrm{d}\bar{g}$ on $SO(n)/M$. The function δ is radial ($SO(n)$-invariant).

Let us compute this function δ. We remark that the matrices

$$A_{pq} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & \cdots & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -1 & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad p < q$$

(1 and -1 being the coefficients of indices (p, q) and (q, p)), form an orthonormal basis of $SO(n) = T_{\mathrm{Id}}SO(n) = T_{\mathrm{o}}(SO(n)/M)$ (o denoting the origin of $SO(n)/M$). Their images tangent at

$$H = \begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & \ddots & \\ & & & h_n \end{pmatrix} \in \Lambda$$

to the orbit through H are

$$[A_{pq}, H] = \frac{h_q - h_p}{\sqrt{2}} \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & \cdots & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad p < q,$$

pairwise orthogonal, of norms $(h_q - h_p)$, which proves that

$$\delta(gHg^{-1}) = \prod_{p<q}(h_q - h_p), \quad g \in SO(n), \quad H \in \Lambda.$$

The Laplace–Beltrami operator L_V on V induces its radial part ΔL_V on Λ, given by [9]:

$$\Delta L_V = \delta^{-1/2} L_\Lambda \delta^{1/2} - \delta^{-1/2} L_\Lambda(\delta^{1/2})$$

where $L_\Lambda = \sum_p \partial^2/\partial h_p^2$ is the Laplace–Beltrami operator on Λ provided with the metric induced by the metric of V.

3. Calogero operator

It has been noticed in [1] that operators of Calogero–Moser–Sutherland type are obtained by conjugation of the radial part of the Laplace–Beltrami operator on a symmetric space by the square root of the density function. Let us make the computation in the case at hand:

$$\delta^{1/2}\Delta L_V \delta^{-1/2} = L_\Lambda - \delta^{-1/2} L_\Lambda(\delta^{1/2}) = \sum_r \frac{\partial^2}{\partial h_r^2} - \delta^{-1/2}\sum_r \frac{\partial^2 \delta^{1/2}}{\partial h_r^2}$$

But

$$\frac{\partial \delta^{1/2}}{\partial h_r} = \frac{1}{2\delta^{1/2}}\frac{\partial \delta}{\partial h_r} = \frac{1}{2\delta^{1/2}}\left(\sum_{p<r}\frac{\delta}{h_r - h_p} - \sum_{q>r}\frac{\delta}{h_q - h_r}\right)$$
$$= \frac{1}{2}\delta^{1/2}\sum_{p\neq r}\frac{1}{h_r - h_p}$$

and

$$\frac{\partial^2 \delta^{1/2}}{\partial h_r^2} = \frac{1}{2}\frac{\partial \delta^{1/2}}{\partial h_r}\sum_{p\neq r}\frac{1}{h_r - h_p} - \frac{1}{2}\delta^{1/2}\sum_{p\neq r}\frac{1}{(h_r - h_p)^2}$$
$$= \frac{1}{4}\delta^{1/2}\left(\sum_{p\neq r}\frac{1}{h_r - h_p}\right)^2 - \frac{1}{2}\delta^{1/2}\sum_{p\neq r}\frac{1}{(h_r - h_p)^2}$$
$$= \frac{1}{4}\delta^{1/2}\left[\sum_{p\neq r}\frac{1}{(h_r - h_p)^2} + \sum_{p\neq r, q\neq r, p\neq q}\frac{1}{(h_r - h_p)(h_r - h_q)}\right]$$
$$- \frac{1}{2}\delta^{1/2}\sum_{p\neq r}\frac{1}{(h_r - h_p)^2}$$
$$= \frac{1}{4}\delta^{1/2}\left[-\sum_{p\neq r}\frac{1}{(h_r - h_p)^2} + \sum_{p\neq r, q\neq r, p\neq q}\frac{1}{(h_r - h_p)(h_r - h_q)}\right].$$

Therefore

$$\delta^{-1/2}\sum_r \frac{\partial^2\delta^{1/2}}{\partial h_r^2} = \frac{1}{4}\left[-\sum_{p\neq r}\frac{1}{(h_r-h_p)^2} + \sum_{p\neq r, q\neq r, p\neq q}\frac{1}{(h_r-h_p)(h_r-h_q)}\right].$$

But the second sum inside the brackets is zero, since:

$$\begin{aligned}
3\cdot &\sum_{p\neq r,q\neq r,p\neq q}\frac{1}{(h_r-h_p)(h_r-h_q)}\\
&= \sum_{p\neq r,q\neq r,p\neq q}\left[\frac{1}{(h_r-h_p)(h_r-h_q)} + \frac{1}{(h_p-h_q)(h_p-h_r)}\right.\\
&\qquad\left. + \frac{1}{(h_q-h_p)(h_q-h_r)}\right]\\
&= \sum_{p\neq r,q\neq r,p\neq q}\left[\frac{h_p-h_q-(h_r-h_q)}{(h_r-h_p)(h_r-h_q)(h_p-h_q)} + \frac{1}{(h_q-h_p)(h_q-h_r)}\right]\\
&= \sum_{p\neq r,q\neq r,p\neq q}\left[\frac{h_p-h_r}{(h_r-h_p)(h_r-h_q)(h_p-h_q)} + \frac{1}{(h_q-h_p)(h_q-h_r)}\right]\\
&= 0.
\end{aligned}$$

Therefore

$$\delta^{1/2}\Delta L_V\delta^{-1/2} = \sum_r \frac{\partial^2}{\partial h_r^2} + \frac{1}{4}\sum_{p\neq r}\frac{1}{(h_r-h_p)^2}.$$

If we apply the same scheme to the $SO(n)$-invariant operator on V:

$$-\frac{1}{2}L_V + \frac{\lambda^2}{2}\|X\|^2 \quad (X\in V),$$

which describes the quantum mechanical harmonic oscillator, we get, after conjugating its radial part, the Calogero operator on Λ with an external quadratic potential:

$$\mathfrak{H} = -\frac{1}{2}\sum_r \frac{\partial^2}{\partial h_r^2} - \frac{1}{8}\sum_{q\neq r}\frac{1}{(h_r-h_q)^2} + \frac{\lambda^2}{2}\sum_r h_r^2 \quad (h_1 < h_2 < \cdots < h_n).$$

4. Propagator

4.1. General case

The propagator of the N-dimensional harmonic oscillator is known. It is given by Mehler's formula (see [8]):

$$K_t(X,X') = (-\mathrm{i})^{N\alpha_t}\mathrm{e}^{-\mathrm{i}N\pi/4}\left|2\pi\frac{\sin\lambda t}{\lambda}\right|^{-N/2}\mathrm{e}^{\mathrm{i}W(X,X')}$$

with

$$W(X,X') = \frac{\lambda}{2\sin\lambda t}[\cos(\lambda t)(\|X\|^2 + \|X'\|^2) - 2\langle X, X'\rangle]$$

and

$$\alpha_t = 0 \quad \text{if } t \in]0, \pi/\lambda[, \quad 1 \ \text{if } t \in]\pi/\lambda, 2\pi/\lambda[, \quad 2 \ \text{if } t \in]2\pi/\lambda, 3\pi/\lambda[, \ldots$$

Let ψ_0 be an initial wave function of the form $\psi_0 = \delta^{-1/2} f_0$ where $f_0 : V \to \mathbb{C}$ is a radial function. Then $\psi_0 \in L^2(V)$ iff $f_{0|\Lambda} \in L^2(\Lambda)$ since

$$\begin{aligned}\int_V |\psi_0(X)|^2 \,\mathrm{d}X &= \int_\Lambda \delta(H) \int_{SO(n)/M} |\psi_0(gHg^{-1})|^2 \,\mathrm{d}\bar{g}\,\mathrm{d}H \\ &= \int_{SO(n)/M} \mathrm{d}\bar{g} \times \int_\Lambda |f_0(H)|^2 \,\mathrm{d}H\end{aligned}$$

(see [9] for this change of variables. $g \in SO(n)$ is a representative of $\bar{g}$).

The wave function at time t is of the form $\psi_t = \delta^{-1/2} f_t$ where f_t is radial, and f is the solution of Schrödinger equation $\mathrm{i}(\partial f/\partial t) = \mathfrak{H} f$ with initial wave function f_0. But for all $H \in \Lambda$:

$$\begin{aligned}&\delta^{-1/2}(H) f_t(H) \\ &= \psi_t(H) = \int_V K_t(H, X')\psi_0(X')\,\mathrm{d}X' \\ &= (-\mathrm{i})^{N\alpha_t} \mathrm{e}^{-\mathrm{i}N\pi/4} \left|2\pi \frac{\sin\lambda t}{\lambda}\right|^{-N/2} \\ &\quad \int_V \mathrm{e}^{\mathrm{i}W(H,X')} \delta^{-1/2}(X') f_0(X')\,\mathrm{d}X', \qquad \text{where } N = \dim V = \frac{n(n+1)}{2} \\ &= (-\mathrm{i})^{(n(n+1)/2)\alpha_t} \mathrm{e}^{-\mathrm{i}n(n+1)\pi/8} \left|2\pi \frac{\sin\lambda t}{\lambda}\right|^{-n(n+1)/4} \\ &\quad \times \int_\Lambda \delta(H')\delta^{-1/2}(H') f_0(H') \int_{SO(n)/M} \mathrm{e}^{\mathrm{i}W(H, gH'g^{-1})}\,\mathrm{d}\bar{g}\,\mathrm{d}H' \\ &= \frac{1}{2^{n-1}} (-\mathrm{i})^{(n(n+1)/2)\alpha_t} \mathrm{e}^{-\mathrm{i}n(n+1)\pi/8} \left|2\pi \frac{\sin\lambda t}{\lambda}\right|^{-n(n+1)/4} \\ &\quad \times \int_\Lambda \delta^{1/2}(H') f_0(H') \int_{SO(n)} \mathrm{e}^{\mathrm{i}W(H, gH'g^{-1})}\,\mathrm{d}g\,\mathrm{d}H'\end{aligned}$$

(since M has cardinal 2^{n-1}).

Hence the propagation of the restriction of f to Λ:

Theorem 1. *The solution of Schrödinger equation* $\mathrm{i}(\partial f/\partial t) = \mathfrak{H} f$ *with initial condition* $f_{|t=0} = f_0 \in L^2(\Lambda)$ *is given by* $f_t(H) = \int_\Lambda k_t(H, H') f_0(H')\,\mathrm{d}H'$ *where*

$$\begin{aligned} & k_t(H, H') \\ & = \frac{1}{2^{n-1}}(-\mathrm{i})^{(n(n+1)/2)\alpha_t}\mathrm{e}^{-\mathrm{i}n(n+1)\pi/8}\left|2\pi\frac{\sin\lambda t}{\lambda}\right|^{-(n(n+1)/4)}\delta^{1/2}(H)\delta^{1/2}(H') \\ & \times\exp\left\{\frac{\mathrm{i}\lambda\cos\lambda t}{2\sin\lambda t}(\|H\|^2+\|H'\|^2)\right\}\int\limits_{SO(n)}\exp\left(-\frac{\mathrm{i}\lambda}{\sin\lambda t}\langle H, gH'g^{-1}\rangle\right)\mathrm{d}g. \end{aligned}$$

Remark. If $\lambda = 0$, the propagator is

$$\begin{aligned} k_t(H, H') = & \frac{1}{2^{n-1}}\mathrm{e}^{-\mathrm{i}n(n+1)\pi/8}(2\pi t)^{-n(n+1)/4}\delta^{1/2}(H)\delta^{1/2}(H') \\ & \times\mathrm{e}^{(\mathrm{i}/2t)(\|H\|^2+\|H'\|^2)}\int\limits_{SO(n)}\exp\left(-\frac{\mathrm{i}}{t}\langle H, gH'g^{-1}\rangle\right)\mathrm{d}g. \end{aligned}$$

In order to compute $\langle H, gH'g^{-1}\rangle$, it is sufficient to know the diagonal entries of $gH'g^{-1}$, given by

$$\begin{aligned} \langle gH'g^{-1}e_p, e_p\rangle = \langle H'g^{-1}e_p, g^{-1}e_p\rangle & = \left\langle\sum_r h'_r(g^{-1})_{rp}e_r, \sum_r(g^{-1})_{rp}e_r\right\rangle \\ & = \sum_r h'_r((g^{-1})_{rp})^2 = \sum_r(g_{pr})^2 h'_r. \end{aligned}$$

Therefore $\langle H, gH'g^{-1}\rangle = \sum_{pq}(g_{pq})^2 h_p h'_q$.

Remark. The propagator is $(2\pi/\lambda)$-periodic if n is congruent to 0 or 3 modulo 4, $(4\pi/\lambda)$-periodic in the remaining cases. This property comes from the fact that the system is the reduction of a $\frac{1}{2}n(n+1)$-dimensional harmonic oscillator. The classical system is $(2\pi/\lambda)$-periodic (Zoll system), for the same reason.

4.2. The case $n = 2$ (two-particle systems)

In the case $n = 2$, we show that the integral in the expression of the propagator can be expressed in terms of a Bessel function.

We parametrize $SO(2)$ in the usual way:

$$g = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$$

and we have

$$\mathrm{d}g = \sqrt{2}\,\mathrm{d}\theta,$$

since the metric chosen on $SO(2)$ is

$$\mathrm{d}s^2 = -\mathrm{tr}\begin{pmatrix} 0 & \mathrm{d}\theta \\ -\mathrm{d}\theta & 0 \end{pmatrix}^2 = 2\,\mathrm{d}\theta^2,$$

and

$$\begin{aligned}\langle H, gH'g^{-1}\rangle &= \cos^2\theta(h_1h_1' + h_2h_2') + \sin^2\theta(h_1h_2' + h_2h_1') \\ &= \tfrac{1}{2}(h_1 + h_2)(h_1' + h_2') + \tfrac{1}{2}(h_2 - h_1)(h_2' - h_1')\cos 2\theta.\end{aligned}$$

Therefore

$$\begin{aligned}&\int\limits_{SO(2)} \exp\left(-\frac{\mathrm{i}\lambda}{\sin\lambda t}\langle H, gH'g^{-1}\rangle\right)\mathrm{d}g \\ &= \exp\left(-\frac{\mathrm{i}\lambda}{2\sin\lambda t}(h_1 + h_2)(h_1' + h_2')\right) \\ &\quad \int\limits_0^{2\pi} \exp\left(-\frac{\mathrm{i}\lambda}{2\sin\lambda t}(h_2 - h_1)(h_2' - h_1')\cos 2\theta\right)\sqrt{2}\,\mathrm{d}\theta \\ &= \sqrt{2}\exp\left(-\frac{\mathrm{i}\lambda}{2\sin\lambda t}(h_1 + h_2)(h_1' + h_2')\right) 2\pi J_0\left(\frac{\lambda(h_2 - h_1)(h_2' - h_1')}{2\sin\lambda t}\right),\end{aligned}$$

where J_0 is the zero order Bessel function. $N = \dim(V)$ being equal to 3, we get the following expression of the propagator:

$$\begin{aligned}&k_t(h_1, h_2; h_1', h_2') \\ &= \frac{1}{2\sqrt{\pi}}\mathrm{i}^{\alpha_t}\mathrm{e}^{-\mathrm{i}3\pi/4}\left|\frac{\sin\lambda t}{\lambda}\right|^{-3/2}\sqrt{(h_2 - h_1)(h_2' - h_1')} \\ &\quad\times \exp\left\{\frac{\mathrm{i}\lambda}{2\sin\lambda t}[(h_1^2 + h_2^2 + h_1'^2 + h_2'^2)\cos\lambda t - (h_2 + h_1)(h_2' + h_1')]\right\} \\ &\quad\times J_0\left(\frac{\lambda(h_2 - h_1)(h_2' - h_1')}{2\sin\lambda t}\right).\end{aligned}$$

Remark. If the system is reduced to its centre of mass, it is equivalent to a (separable) system of two uncoupled particles on a line, of equal masses, independent from one another, one in the harmonic potential $\frac{1}{2}\lambda^2x^2$, the other in the potential $\frac{1}{2}\lambda^2x^2 - 1/8x^2$. Our result for two-particle sytems can thus be deduced from the results of Khandekar and Lawande [11]. The reader can also refer to Schulman's book [18] which gives a list of computable propagators.

5. Stationary phase approximation

In view of the expression of the propagator found in the case $n = 2$, which contains a Bessel function, the semi-classical approximation of the propagator is not exact for finite times. Yet, it is possible to obtain the asymptotic behaviour of the propagator for short times, using the stationary phase method.

According to this method,

$$(2\pi t)^{-(1/2)\dim(SO(n)/M)} \int_{SO(n)/M} \exp\left(-\frac{\mathrm{i}\lambda}{\sin\lambda t}\langle H, gH'g^{-1}\rangle\right) \mathrm{d}\bar{g}$$

$$= \sum_{\bar{g}} c(\bar{g}) \exp\left(-\frac{\mathrm{i}}{t}\langle H, gH'g^{-1}\rangle\right) + \mathrm{O}(t),$$

the summation being made on all the critical points of

$$f : \bar{g} \in SO(n)/M \mapsto -\langle H, gH'g^{-1}\rangle,$$

$c(\bar{g})$ being equal to

$$\exp\left(\mathrm{i}\frac{\pi}{4}\operatorname{sgn}\operatorname{Hess}_{\bar{g}} f\right) \mid \det \operatorname{Hess}_{\bar{g}} f \mid^{-1/2}.$$

Here $\operatorname{Hess}_{\bar{g}} f$ denotes the Hessian matrix of f at $\bar{g}$ in a basis of $T_{\bar{g}}(SO(n)/M)$ of volume 1, and sgn its signature, i.e. the difference between the numbers of its positive and negative eigenvalues.

Let $H, H' \in A$. The critical points of the function f are those for which H is orthogonal to the orbit through H' at $gH'g^{-1}$, which amounts to $gH'g^{-1}$ being diagonal. Therefore the critical points of $\bar{g}$ are in a one to one correspondence with the matrices obtained by permutation of the diagonal entries of H'. Let $\bar{g}$ be a critical point, $H'' = gH'g^{-1}$. The vectors $A_{pq} \cdot g$ $(p < q)$ form an orthonormal basis of $T_g SO(n)$ and

$$\begin{aligned}
&\langle H, \exp X gH'g^{-1} \exp(-X)\rangle \\
&\quad = \langle H, \exp X H'' \exp(-X)\rangle \\
&\quad = \langle H, (I + X + \tfrac{1}{2}X^2)H''(I - X + \tfrac{1}{2}X^2)\rangle + \mathrm{o}(\|X\|^2) \\
&\quad = \langle H, H'' + [X, H''] + \tfrac{1}{2}(X^2H'' + H''X^2) - XH''X\rangle + \mathrm{o}(\|X\|^2) \\
&\quad = \langle H, H''\rangle + \tfrac{1}{2}\langle H, X^2H'' + H''X^2 - 2XH''X\rangle + \mathrm{o}(\|X\|^2).
\end{aligned}$$

If $X = \sum_{p<q} s_{pq} A_{pq}$, we get

$$\begin{aligned}
&-\langle H, \exp X gH'g^{-1} \exp(-X)\rangle \\
&\quad = -\langle H, H''\rangle + \frac{1}{2}\sum_{p<q}(h_q - h_p)(h''_q - h''_p)s_{pq}^2.
\end{aligned}$$

The Hessian matrix of f at $\bar{g}$ is thus diagonal. The absolute value of its determinant is $\delta(H)\delta(H')$. Its signature is $\frac{1}{2}n(n-1) - 2I(\sigma)$, where $\sigma \in S_n$ is the permutation of the set $\{1, \ldots, n\}$ corresponding to $\bar{g}$ and $I(\sigma) = \operatorname{Card}\{(p, q) \mid p < q, \sigma(p) > \sigma(q)\}$ is the number of inversions in σ.

Thus

$$\int\limits_{SO(n)/M} \exp\left(-\frac{\mathrm{i}\lambda}{\sin\lambda t}\langle H, gH'g^{-1}\rangle\right) \mathrm{d}\bar{g}$$

$$= (2\pi t)^{n(n-1)/4} \left[\delta^{-1/2}(H)\delta^{-1/2}(H')\mathrm{e}^{\mathrm{i}(n(n-1)/8)\pi} \right.$$

$$\left. \times \sum_{\sigma\in S_n} \mathrm{e}^{-\mathrm{i}(\pi/2)I(\sigma)}\mathrm{e}^{-(\mathrm{i}/t)(h_1h'_{\sigma(1)}+\cdots+h_nh'_{\sigma(n)})} + \mathrm{O}(t) \right].$$

Theorem 2. *The short time approximation of the propagator is given by*

$$k_t(H, H') = (2\pi t)^{-n/2}\mathrm{e}^{-\mathrm{i}n\pi/4} \left[\sum_{\sigma\in S_n} \mathrm{e}^{-\mathrm{i}(\pi/2)I(\sigma)}\mathrm{e}^{(\mathrm{i}/2t)\|H-\sigma^{-1}(H')\|^2} + \mathrm{O}(t) \right],$$

where $I(\sigma) = I(\sigma^{-1})$ is the number of infinite potential wells $h_i = h_j$ across the segment $[H, \sigma^{-1}(H')]$.

The contribution of H' to the propagator for short times can be interpreted as resulting of broken straight lines from H' to H, with a phase shift of $-\frac{1}{2}\pi$ at each "reflexion" on a potential well, i.e. at each collision between two particles.

Remark. By computing as in the preceding section the propagator for functions on $u(n)$ invariants for the adjoint action of $U(n)$, we can get the Fourier transform of the delta distribution of an adjoint orbit, a result first proved by Harish Chandra (cf. [8]).

As a manifold transversal to the orbits, we chose the set A of diagonal purely imaginary matrices whose diagonal entries: $\mathrm{i}h_1, \ldots, \mathrm{i}h_n$ verify $h_1 < h_2 < \cdots < h_n$. The adjoint orbits are diffeomorphic to $U(n)/M$, M being the isotropy subgroup of purely imaginary diagonal matrices, i.e. the set of diagonal matrices in $U(n)$. The radial part ΔL of the Laplace–Beltrami operator L on the Euclidean space $u(n)$ displays

$$\delta^{1/2}\Delta L \circ \delta^{-1/2} = \sum_r \frac{\partial^2}{\partial h_r^2},$$

where $\delta = \prod_{r<l}(h_l - h_r)^2$ is the density function, i.e. $\mathrm{d}X = \delta \cdot \mathrm{d}h_1 \cdots \mathrm{d}h_n \,\mathrm{d}\bar{g}$, $\mathrm{d}\bar{g}$ denoting the Riemannian measure on $U(n)/M$.

The propagator of Shrödinger equation $\mathrm{i}(\partial/\partial t) = -\frac{1}{2}L$ for a free particle on $u(n)$ is

$$K_t(X, X') = \mathrm{e}^{-\mathrm{i}N\pi/4}\frac{1}{(2\pi t)^{N/2}}\mathrm{e}^{\mathrm{i}W(X,X')},$$

where

$$W(X, X') = \frac{1}{2t}(\|X\|^2 + \|X'\|^2 - 2\langle X, X'\rangle)$$

and $N = \dim u(n) = n^2$.

Let h denote the set of diagonal purely imaginary matrices. If a family of radial functions $(\psi_t)_{t\geq 0}$ in $L^2(u(n))$ verifies Schrödinger equation $\mathrm{i}(\partial/\partial t) = -\frac{1}{2}L$, the family of functions $(f_t)_{t\geq 0}$ in $L^2(h)$, skew-symmetric in the variables $h_1, \dots, h_n$, defined by $f_t(H) = \delta^{1/2}\psi_t(H) = \prod_{r<l}(h_l - h_r)\psi_t(H)$ verifies the equation $\mathrm{i}(\partial/\partial t) = -\frac{1}{2}\sum_r \partial^2/\partial h_r^2$. Hence f_t is propagated by the kernel

$$k_t(H, H') = \mathrm{e}^{-\mathrm{i}n\pi/4}\frac{1}{(2\pi t)^{n/2}}\exp\left\{\frac{\mathrm{i}}{2t}(\|H\|^2 + \|H'\|^2 - 2\langle H, H'\rangle)\right\}$$

and its restriction to Λ by

$$\sum_{\sigma\in S_n}\epsilon(\sigma)k_t(H, \sigma(H')),$$

and also by

$$\delta^{1/2}(H)\delta^{1/2}(H')\int_{U(n)/M} K_t(H, gH'g^{-1})\,\mathrm{d}\bar{g}.$$

We deduce Harish Chandra's formula for the Fourier transform of the delta distribution of an adjoint orbit

$$\int_{U(n)/M}\exp(-\mathrm{i}\langle H, gH'g^{-1}\rangle)\,\mathrm{d}\bar{g} = \frac{\mathrm{e}^{\mathrm{i}n(n-1)\pi/4}(2\pi)^{n(n-1)/2}}{\prod_{r<l}[(h_l - h_r)(h'_l - h'_r)]}\sum_{\sigma\in S_n}\epsilon(\sigma)\exp(-\mathrm{i}\langle H, \sigma(H')\rangle),$$

which can be interpreted as a case of exactitude of the stationary phase approximation [7].

6. Other interaction constants

Here we investigate two different schemes to obtain other interaction constants than $-\frac{1}{4}$.

6.1. Eigenfunctions of orbital Laplace–Beltrami operators

Using the separation of variables, we consider functions which are the product of a radial function and a fixed angular function whose restriction to each orbit is an eigenfunction of the orbital Laplace–Beltrami operator.

For this purpose, we need the expression of the Laplace–Beltrami operator L_V with respect to the coordinates $(H, \bar{g}) \in \Lambda \times SO(n)/M$.

L_V splits into a transversal and an orbital part (cf. [9]), which means that at each point X of V, for any function $\psi \in C^\infty(V)$,

$$(L_V\psi)(X) = ((L_V)_T\psi)(X) + (L_{O_X}\psi_{|O_X})(X),$$

where $(L_V)_T$ is the transversal part of L_V:

$$((L_V)_T\psi)(gHg^{-1}) = [(\Delta L_V)(H' \in \Lambda \mapsto \psi(gH'g^{-1})](H),$$

and L_{O_X} is the Laplace–Beltrami operator on the orbit O_X through X.

Let $H \in \Lambda$. Let us denote by A_{pq} the infinitesimal rotations defined in Section 2, as well as the vector fields that they induce on O_H. The vectors $B_{pq} = A_{pq}/(h_q - h_p),\ p < q$, form at each point of the orbit an orthonormal basis of the tangent space. For every function f defined on O_H, the gradient of f is given by $\mathrm{grad}(f) = \sum_{p<q}(B_{pq}f)B_{pq}$. Furthermore, the metric of the orbit and therefore its volume form being invariant under the action of $SO(n)$, the vector fields B_{pq} are divergence free. Hence

$$\begin{aligned} L_{O_X}f = \mathrm{div}\ \mathrm{grad} f &= \sum_{p<q}[\langle \mathrm{grad}(B_{pq}f), B_{pq}\rangle + (B_{pq}f)\mathrm{div}(B_{pq})] \\ &= \sum_{p<q} B_{pq}^2 f = \sum_{p<q} \frac{1}{(h_q - h_p)^2} A_{pq}^2 f. \end{aligned}$$

Let c be a function on $SO(n)/M$ inducing on each orbit an eigenfunction of the orbital Laplace–Beltrami operator. Functions of the form $\psi(gHg^{-1}) = f(H)c(\bar{g})$ conserve this form through their time evolution with respect to the harmonic oscillator problem.

The orbital Laplace–Beltrami operator is a linear combination of the squares of the infinitesimal rotations A_{pq}, $p < q$, in the coordinates planes. Therefore, if $c(\bar{g})$ is an eigenfunction of the square of the infinitesimal rotation in each coordinates plane, then it will satisfy to the condition above .

Let χ be a representation of $SO(n)$ on some vector space E. If $u \in E$ is an eigenvector for each operator $(\mathrm{d}\chi A_{pq})^2$, and η any linear form on E, then the representation coefficient $c(g) = \eta(\chi(g)u)$ is also an eigenvector of the A_{pq}^2's with the same eigenvalues. Indeed

$$\begin{aligned} A_{pq}^2 c(g) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}_{|t=0} c(g \exp(tA_{pq})) \\ &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}_{|t=0} \eta(\chi(g \exp(tA_{pq})u) \\ &= \eta(\chi(g)(\mathrm{d}\chi A_{pq})^2 u). \end{aligned}$$

Let us consider the irreducible representation χ of $SO(n)$ on the space E of harmonic polynomials of n indeterminates $x_1, \ldots, x_n$ homogeneous of degree n (cf. [21]). We have

$$\mathrm{d}\chi(A_{pq}) = \frac{1}{\sqrt{2}}\left(x_q \frac{\partial}{\partial x_p} - x_p \frac{\partial}{\partial x_q}\right).$$

Note that the harmonic monomial $u = x_1 x_2 \cdots x_n$ yields

$$\mathrm{d}\chi(A_{pq})u = \frac{1}{\sqrt{2}}(x_1 \cdots \hat{x}_p \cdots x_q^2 \cdots x_n - x_1 \cdots x_p^2 \cdots \hat{x}_q \cdots x_n)$$

(the symbol ^ designing an omitted factor) and

$$(\mathrm{d}\chi(A_{pq}))^2 u = -2u.$$

Furthermore u is left invariant by the rotations of angle π in the coordinates planes (which amount to turning two of the coordinates x_p and x_q into their opposite), thus the coefficients $c(g) = \eta(\chi(g)u)$ define functions $c(\bar{g})$ on the quotient space $SO(n)/M$ and therefore on the orbits. For instance, if η is the component on the vector $x_1 x_2 \cdots x_n$ of the canonical basis of E, $c(\bar{g}) = \sum_{\sigma \in S_n} g_{1\sigma(1)} g_{2\sigma(2)} \cdots g_{n\sigma(n)}$ where $g = ((g_{pg})) \in SO(n)$ is a representative of $\bar{g}$. These functions are eigenfunctions, and the associated eigenvalue on the orbit O_H is

$$-2 \sum_{p<q} \frac{1}{(h_q - h_p)^2}.$$

Remark. These functions are exceptional. The author has verified that in the case $n = 3$, there is no other non-proportional function which is an eigenfunction for each $(\mathrm{d}\chi A_{pq})^2$.

Therefore, for $\psi : gHg^{-1} \mapsto f(H)c(\bar{g})$,

$$L_V(\delta^{1/2}\psi)(gHg^{-1}) = \delta^{1/2}(H) \left[\sum_r \frac{\partial^2}{\partial h_r^2} - \frac{3}{2} \sum_{p<r} \frac{1}{(h_r - h_p)^2} \right] (f)(H)c(\bar{g})$$

Hence we have:

Theorem 3. *The propagator of the Hamilton operator*

$$-\frac{1}{2} \sum_r \frac{\partial^2}{\partial h_r^2} + \frac{3}{4} \sum_{p<r} \frac{1}{(h_r - h_p)^2} + \frac{\lambda^2}{2} \|H\|^2$$

is

$$\begin{aligned} k_t(H, H') = {} & \frac{1}{2^{n-1}} (-\mathrm{i})^{(n(n+1)/2)\alpha_t} \mathrm{e}^{-\mathrm{i}n(n+1)\pi/8} \left| 2\pi \frac{\sin \lambda t}{\lambda} \right|^{-n(n+1)/4} \\ & \times \delta^{1/2}(H) \delta^{1/2}(H') \\ & \times \frac{1}{c(1)} \exp\left(\frac{\mathrm{i}\lambda \cos \lambda t}{2 \sin \lambda t} (\|H\|^2 + \|H'\|^2) \right) \\ & \int_{SO(n)} \exp\left(-\frac{\mathrm{i}\lambda}{\sin \lambda t} \langle H, gH'g^{-1} \rangle c(g) \right) \mathrm{d}g. \end{aligned}$$

In the exceptional case $n = 2$, all the irreducible representations of $SO(2)$ are unidimensional and each vector of the representation line is obviously an eigenvector for $A_{12} = (1/\sqrt{2})(\mathrm{d}/\mathrm{d}\theta)$. If $c(\bar{g}_\theta) = \mathrm{e}^{2\mathrm{i}n\theta}$, g_θ designing the rotation of angle θ and n a fixed integer ($c(\bar{g})$ is not single-valued for n half integer), $A_{pq}^2 c = -n^2 c$.

Hence we can obtain all the interaction constants of the form $n^2 - \frac{1}{4}$, $n \in \mathbb{N}$. The integral in the propagator can be expressed using a Bessel function of order n:

$$k_t(h_1, h_2; h'_1, h'_2)$$
$$= \frac{1}{2\sqrt{\pi}} \mathrm{i}^{\alpha_t - n} \mathrm{e}^{-\mathrm{i}3\pi/4} \left| \frac{\sin \lambda t}{\lambda} \right|^{-3/2} \sqrt{(h_2 - h_1)(h'_2 - h'_1)}$$
$$\times \exp \left\{ \frac{\mathrm{i}\lambda}{2 \sin \lambda t} [(h_1^2 + h_2^2 + h_1'^2 + h_2'^2) \cos \lambda t - (h_2 + h_1)(h'_2 + h'_1)] \right\}$$
$$J_n \left(\frac{\lambda}{2 \sin \lambda t} (h_2 - h_1)(h'_2 - h'_1) \right).$$

6.2. Other symmetric spaces

Olshaneski and Perelomov remarked in [15] that the radial parts of the Laplace–Beltrami operator of symmetric spaces $SU(2n)/Sp(n)$ and E_6/F_4 are conjugated to Sutherland systems with interaction constants 2 and 12. The corresponding Calogero systems can be obtained by letting the curvature go to zero with homothetical transformations and by making a central extension to get rid of the constraint "$\sum h_p = 0$". For instance, when the interaction constant is 2, we consider the action of $Sp(n)$ by conjugacy on the orthogonal complement of $sp(n)$ in $u(2n)$, which is the space of matrices of the form

$$\begin{pmatrix} Z_1 & Z_2 \\ \bar{Z}_2 & -\bar{Z}_1 \end{pmatrix}, \quad Z_1 \in u(n),\ Z_2 \text{ complex skew-symmetric.}$$

7. Energy levels and trace of the propagator

Let us denote by $X = ((X_{pq}))$ the generic element of V and set

$$z_{pp} = X_{pp}, \qquad z_{pq} = \frac{1}{\sqrt{2}} X_{pq}, \quad p < q.$$

The z_{pq}, $p \le q$, form a system of orthonormal coordinates on V.

The one-dimensional harmonic oscillator with Hamilton operator $-\frac{1}{2}(\mathrm{d}^2/\mathrm{d}x^2) + \frac{1}{2}\lambda^2 x^2$ has eigenfunctions of the form $\psi(x) = \text{cste} \cdot \mathrm{e}^{-\lambda x^2/2} H_p(\sqrt{\lambda} x)$ (where H_p denotes the pth Hermite polynomial) with eigenvalues $(p + \frac{1}{2})\lambda$ (see [12] for instance). The eigenfunctions of the harmonic oscillator for which Hamiltonian is

$$-\frac{1}{2} L_V + \frac{\lambda^2}{2} \|X\|^2$$

are thus the functions of the form

$$\psi(X) = \exp\left(-\lambda \frac{\|X\|^2}{2}\right) \sum_{\left\{ (a_{pq})_{p \le q} \in \mathbb{N}^{n(n+1)/2},\ \sum_{p \le q} a_{pq} = r \right\}} c_{a_{11} a_{12} \cdots a_{nn}} \prod_{p \le q} H_{a_{pq}}(\sqrt{\lambda} z_{pq}),$$

where the $c_{a_{11} a_{12} \cdots a_{nn}}$ are arbitrary constants. The corresponding energy level is $(r + N/2)\lambda = [r + n(n+1)/4]\lambda$.

The eigenfunctions of the reduced operator are obtained by multiplicating by the square root of the density function the restriction to Λ of the radial functions among the ones above.

Therefore each eigenfunction of the reduced system is of the form

$$\varphi(H) = \sqrt{\delta(H)}\exp\left(-\lambda\frac{\|H\|^2}{2}\right)P(h_1,\ldots,h_n),$$

where P is a polynomial.

Furthermore, since the invariant function $\psi(gHg^{-1}) = \exp(-\lambda\|H\|^2/2)P(h_1,\ldots,h_n)$ defined by φ is an eigenfunction of the harmonic oscillator, it is analytic and thus

$$\psi\begin{pmatrix} h_{\sigma(1)} & & \\ & \ddots & \\ & & h_{\sigma(n)}\end{pmatrix} = \exp\left(-\lambda\frac{\|H\|^2}{2}\right)P(h_{\sigma(1)},\ldots,h_{\sigma(n)})$$

for every permutation $\sigma \in S(n)$.

But

$$\begin{pmatrix} h_{\sigma(1)} & & \\ & \ddots & \\ & & h_{\sigma(n)}\end{pmatrix}$$

is conjugated to

$$\begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_n\end{pmatrix},$$

hence the left-hand side of the equality above is also equal to $\exp(-\lambda\|H\|^2/2)P(h_1,\ldots, h_n)$, which proves that the polynomial P is symmetric.

We will prove that the functions of the form $\varphi(H) = \delta^a(H)\exp(-\lambda\|H\|^2/2)P(H)$, where P is a homogeneous symmetric polynomial, provide a basis of $L^2(\Lambda)$ in which the matrix of $\mathfrak{H} = -\frac{1}{2}\sum_r \partial^2/\partial h_r^2 + b\sum_{q\neq r} 1/(h_r-h_q)^2 + \frac{1}{2}\lambda^2\sum_r h_r^2$ is triangular. The exponent $a > 0$ will depend on the interaction constant b and be equal to $\frac{1}{2}$ in the case of the reduction of radial functions ($b = -1/8$).

We have

$$\frac{\partial\varphi}{\partial h_r} = \delta^a(H)\exp\left(-\lambda\frac{\|H\|^2}{2}\right)\left[a\sum_{p\neq r}\frac{1}{(h_r-h_p)}P - \lambda h_r P + \frac{\partial P}{\partial h_r}\right],$$

$$\begin{aligned}\frac{\partial^2\varphi}{\partial h_r^2} &= \delta^a(H)\exp\left(-\lambda\frac{\|H\|^2}{2}\right)\\ &\quad\times\left[\left(\sum_{p\neq r}\frac{a^2-a}{(h_r-h_p)^2} + \sum_{p\neq r, q\neq r, p\neq q}\frac{a^2}{(h_r-h_p)(h_r-h_q)}\right)P\right.\\ &\quad + (\lambda^2h_r^2-\lambda)P + \frac{\partial^2 P}{\partial h_r^2} + 2\left(a\sum_{p\neq r}\frac{1}{h_r-h_p} - \lambda h_r\right)\frac{\partial P}{\partial h_r}\\ &\quad\left. + 2a\sum_{p\neq r}\frac{1}{h_r-h_p}(-\lambda h_r)P\right].\end{aligned}$$

Since

$$\sum_r \sum_{p \neq r} \frac{h_r}{h_r - h_p} = \frac{n(n-1)}{2}$$

and

$$\sum_{p \neq r, q \neq r, p \neq q} \frac{1}{(h_r - h_p)(h_r - h_q)} = 0$$

(cf. Section 3),

$$\mathfrak{H}\varphi = \delta^a(H) \exp\left(-\lambda \frac{\|H\|^2}{2}\right)$$
$$\times \left[-\frac{1}{2} \sum_r \frac{\partial^2 P}{\partial h_r^2} + \sum_r \left(\lambda h_r - a \sum_{p \neq r} \frac{1}{h_r - h_p} \right) \frac{\partial P}{\partial h_r} + \left(a \frac{n(n-1)}{2} + \frac{n}{2} \right) \lambda P \right],$$

where $a > 0$ is chosen so that $\frac{1}{2}(a^2 - a)$ is equal to the interaction constant b.

If P is homogeneous of degree d, then Euler formula $\sum_r h_r(\partial P / \partial h_r) = d \cdot P$ yields:

$$\mathfrak{H}\varphi = \delta^a(H) \exp\left(-\lambda \frac{\|H\|^2}{2}\right)$$
$$\times \left[-\frac{1}{2} \sum_r \frac{\partial^2 P}{\partial h_r^2} - a \sum_r \sum_{p \neq r} \frac{1}{h_r - h_p} \frac{\partial P}{\partial h_r} + \left(a \frac{n(n-1)}{2} + \frac{n}{2} + d \right) \lambda P \right].$$

If moreover P is symmetric, then $\partial P / \partial h_r - \partial P / \partial h_p$ is zero when $h_r = h_p$ and thus is divisible by $(h_r - h_p)$.

Therefore, we obtain

$$\mathfrak{H}\varphi = \left(\frac{n(an + 1 - a)}{2} + d \right) \lambda \varphi + \delta^a(H) \exp\left(-\lambda \frac{\|H\|^2}{2}\right) Q,$$

where Q is a symmetric polynomial of degree at most $d - 2$.

Hence we have:

Theorem 4. *Each eigenfunction corresponds to a symmetric polynomial, the energy levels are* $\frac{1}{2}(n(an + 1 - a) + d)\lambda$ $(d \in \mathbb{N})$, *with multiplicity equal to the dimension of the space of homogeneous symmetric polynomials of* n *indeterminates of degree* d. *This dimension is the cardinal of the set* $\{(a_1, \ldots, a_n) \in \mathbb{N}^n \mid a_1 \leq a_2 \leq \cdots \leq a_n,\ a_1 + \cdots + a_n = d\}$.

We refer the reader to [16] for an anologue of this theorem for more general cases.

Hence the trace of the propagation operator has a very simple closed form expression:

$$tr(e^{-\mathrm{i}t\mathfrak{H}}) = \sum_{E \text{ energy level}} \exp(-\mathrm{i}Et) \quad (\operatorname{Im} t < 0)$$
$$= \exp\left(-\mathrm{i}\frac{n(an+1-a)\lambda t}{2}\right) \sum_{a_1=0}^{\infty} \exp(-ia_1\lambda t)$$
$$\times \sum_{a_2=a_1}^{\infty} \exp(-\mathrm{i}a_2\lambda t) \cdots \sum_{a_n=a_{n-1}}^{\infty} \exp(-\mathrm{i}a_n\lambda t).$$

But

$$\sum_{a_n=a_{n-1}}^{\infty} \exp(-\mathrm{i}a_n\lambda t) = \exp(-\mathrm{i}a_{n-1}\lambda t) \sum_{a_n'=0}^{\infty} \exp(-\mathrm{i}a_n'\lambda t)$$
$$= \exp(-\mathrm{i}a_{n-1}\lambda t)\frac{1}{1-\exp(-\mathrm{i}\lambda t)}$$
$$= \exp(-\mathrm{i}a_{n-1}\lambda t)\frac{\exp(\mathrm{i}\lambda t/2)}{2i\sin(\lambda t/2)}.$$

Thus the trace of the propagation operator $tr(\mathrm{e}^{-\mathrm{i}t\mathfrak{H}})$ is equal to

$$\exp\left(-\mathrm{i}\frac{n(an+1-a)\lambda t}{2}\right)\frac{\exp(\mathrm{i}\lambda t/2)}{2\mathrm{i}\sin(\lambda t/2)} \sum_{a_1=0}^{\infty} \exp(-ia_1\lambda t)$$
$$\times \sum_{a_2=a_1}^{\infty} \exp(-\mathrm{i}a_2\lambda t) \cdots \sum_{a_{n-1}=a_{n-2}}^{\infty} \exp(-2\mathrm{i}a_{n-1}\lambda t)$$
$$= \exp\left(-\mathrm{i}\frac{n(an+1-a)\lambda t}{2}\right)\frac{\exp(\mathrm{i}\lambda t/2)}{2\mathrm{i}\sin(\lambda t/2)}\frac{\exp(\mathrm{i}\lambda t)}{2\mathrm{i}\sin(\lambda t)}\frac{\exp(\mathrm{i}3\lambda t/2)}{2\mathrm{i}\sin(3\lambda t/2)}$$
$$\times \cdots \frac{\exp(\mathrm{i}n\lambda t/2)}{2\mathrm{i}\sin(n\lambda t/2)}.$$

Theorem 5. *The trace of the propagation operator* $tr(\mathrm{e}^{-\mathrm{i}t\mathfrak{H}})$ *is equal to*

$$\frac{\exp(i(1-2a)n(n-1)\lambda t/4)}{(2\mathrm{i})^n \sin(\lambda t/2)\sin(\lambda t)\sin(3\lambda t/2)\cdots\sin(n\lambda t/2)}.$$

Besides, the trace can also be expressed with respect to the propagator:

$$tr(\mathrm{e}^{-\mathrm{i}t\mathfrak{H}}) = \int_{\Lambda} k_t(H, H)\,\mathrm{d}H.$$

In the case of two-particle systems, we obtain known identities involving Bessel functions, connected with the formula (cf. [22])

$$\int_0^{+\infty} e^{-ax} J_\nu(bx)\,dx = \frac{[\sqrt{a^2+b^2}-a]^\nu}{b^\nu\sqrt{a^2+b^2}}.$$

More complex identities may be obtained by considering systems with three particles or more.

8. Bosons and fermions

The propagators computed so far allow to describe the motion of n particles whichever statistic (bosonic or fermionic) the particles obey.

The wave function f_t at time t of such a system is not defined on Λ, but on the space of diagonal matrices with distinct entries:

$$\tilde{\Lambda} = \bigcup_{\sigma\in S_n} \sigma(\Lambda),$$

where a permutation $\sigma \in S_n$ of $\{1, 2, \ldots, n\}$ acts on the space of diagonal matrices by

$$\sigma\begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & \ddots & \\ & & & h_n \end{pmatrix} = \begin{pmatrix} h_{\sigma^{-1}(1)} & & & \\ & h_{\sigma^{-1}(2)} & & \\ & & \ddots & \\ & & & h_{\sigma^{-1}(n)} \end{pmatrix}.$$

But $\tilde{f}_t$ is determined by its restriction to f_t at Λ:

$$\tilde{f}_t(\sigma(H)) = f_t(H) \quad \text{for bosons,}$$
$$\tilde{f}_t(\sigma(H)) = \epsilon(\sigma) f_t(H) \quad \text{for fermions,}$$
$$H \in \Lambda, \quad \sigma \in S_n,$$

$\epsilon(\sigma)$ denotes the signature of σ.

In both cases, the propagation of $\tilde{f}_t$ is given by the one of f_t: the solution of the Schrödinger equation

$$i\frac{\partial \tilde{f}_t}{\partial t} = -\frac{1}{2}\Delta\tilde{f}_t + \left(a\sum_{p\neq q}\frac{1}{(h_q-h_p)^2} + b\sum_p h_p^2\right)\tilde{f}_t,$$

with initial wave function, $\tilde{f}_0$, is given by

$$\tilde{f}_t(\sigma(H)) = \int_\Lambda k_t(H, H') f_0(H')\,dH' \quad \text{(for bosons),}$$

or

$$\epsilon(\sigma)\int_\Lambda k_t(H, H') f_0(H')\,dH' \quad \text{(for fermions)}$$

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